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Two-fermion bound state in a Bose-Einstein condensate

Weiping Zhang, Han Pu, Chris P. Search, Pierre Meystre and E. M. Wright
Optical Sciences Center, The University of Arizona, Tucson, AZ 85721
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A nonlinear Schrödinger equation is derived for the dynamics of a beam of ultracold fermionic atoms traversing a Bose-Einstein condensate. The condensate phonon modes are shown to provide a nonlinear medium for the fermionic atoms. A two-fermion bound state is predicted to arise, and the signature of the bound state in a nonlinear atom optics experiment is discussed.

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The recent experimental success in creating quantum-degenerate atomic Fermi gases [1, 2, 3, 4, 5] is opening up fascinating new opportunities to explore the quantum statistics of ultracold atoms. Of particular interest in this context is the formation of atomic Cooper pairs in ultracold Fermi gases [6, 7, 8, 9, 10, 11, 12]. Another important question is whether nonlinear atom optical effects, which are now well established in bosonic systems, can also occur with fermions. Fermionic behavior is strongly constrained by the Pauli exclusion principle. This limits the variety of possible nonlinear atom optics effects, but also offers the potential for novel applications without analogs in optics. These include, for example, low-noise inertial and rotation sensors, and quantum information processing. Research on nonlinear atom optical effects in a quantum-degenerate Fermi gas has recently been initiated. Two theoretical papers [13, 14] have indicated that fermionic atom four-wave mixing is possible under appropriate conditions.

Our goal in this Letter is to explore a new situation where a gas of bosons serves as a nonlinear medium for fermionic atoms. In particular, we study how the interatomic interaction between a Bose-Einstein condensate and a fermionic beam can be employed to manipulate the quantum state of the beam. By drawing on the analogy to nonlinear optics we describe the interaction in terms of an effective attractive Kerr nonlinearity, and show that a two-fermion bound state can result with a unique signature in a nonlinear atom optical experiment.

Since the Pauli exclusion principle precludes the direct evaporative cooling of spin-polarized fermionic samples, current experiments employ either unpolarized fermionic mixtures [1, 4] or Bose-Fermi mixtures [2, 3, 5]. Our starting point is a beam of fermionic atoms with two internal spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ interacting via two-body collisions with a quantum degenerate Bose gas. The atomic Bose field is decomposed in the familiar way as $\hat\psi_B(\mathbf{r}) = \phi_B(\mathbf{r}) + \hat\xi(\mathbf{r}) = \sqrt{n_B(\mathbf{r})} + \hat\xi(\mathbf{r})$, where $\phi_B(\mathbf{r})$ is the condensate wave function, taken to be real for simplicity, $\hat\xi(\mathbf{r})$ describes the elementary excitations above the condensate, and $n_B = |\phi_B|^2$ is the condensate density. The fermionic field is described by the field operators $\hat\psi_\sigma(\mathbf{r})$, with $\sigma = \{\uparrow, \downarrow\}$. In the following, it will also be useful to introduce the density fluctuations of the fermion fields by $\delta\hat n_\sigma = \hat\psi^\dagger_\sigma(\mathbf{r})\hat\psi_\sigma(\mathbf{r}) - \langle\hat\psi^\dagger_\sigma(\mathbf{r})\hat\psi_\sigma(\mathbf{r})\rangle = \hat n_\sigma - n_\sigma$.

The Bose and Fermi systems are coupled by two-body interactions. In the shapeless approximation, and taking into account that s-wave scattering is forbidden between fermionic atoms of same spin, we find that to lowest-order in $\hat\xi(\mathbf{r})$, the dynamics of the bosonic atoms is given by the coupled equations [11]

$$\left(\hat H_B^{(0)} + g n_B(\mathbf{r}) + \sum_\sigma f_\sigma n_\sigma(\mathbf{r})\right)\phi_B = \mu_B\phi_B,$$
$$i\hbar\frac{\partial\hat\xi}{\partial t} = \left(\hat H_B - \mu_B\right)\hat\xi + g\phi^2\hat\xi^\dagger + \phi\sum_\sigma f_\sigma\delta\hat n_\sigma. \quad (1)$$

To the same order, the fermionic field equations are

$$i\hbar\frac{\partial\hat\psi_\sigma}{\partial t} = \left(\hat H_{F\sigma} - \mu_\sigma\right)\hat\psi_\sigma$$
$$+\ f_\sigma(\phi_B\hat\xi^\dagger + \phi_B^*\hat\xi)\hat\psi_\sigma + h\hat\psi^\dagger_{\sigma'}\hat\psi_{\sigma'}\hat\psi_\sigma. \quad (2)$$

Here, $\hat H_\alpha^{(0)} = \hat T_\alpha + V_\alpha(\alpha = B, \sigma)$ are the single-particle Hamiltonians for bosonic atoms and for fermionic atoms of spin σ, respectively, $\hat T_\alpha$ and V_α are the associated kinetic energy and trapping potential. The Hamiltonians $\hat H_B = \hat H_B^{(0)} + 2gn_B + \sum_\sigma f_\sigma n_\sigma$, and $\hat H_{F\sigma} = \hat H_{F\sigma}^{(0)} + f_\sigma n_B$, also include the self-contribution to the mean-field energy of the respective fields. Finally, μ_α are the chemical potentials, and the parameters g, f and h represent the boson-boson, boson-fermion, and fermion-fermion interaction strengths. In terms of the s-wave scattering lengths a, they are given by $g = 4\pi\hbar^2 a_B/m_B, f_\sigma = 2\pi\hbar^2 a_{BF\sigma}/m_r, h = 4\pi\hbar^2 a_{\uparrow\downarrow}/m_F$, with $m_r = m_F m_B/(m_F + m_B)$ being the reduced mass.

In previous work [11] we discussed how the coupling to a fermionic component can change the dynamical stability of a Bose condensate. Here we concentrate instead on how the presence of a condensate can induce nonlinear dynamics of a fermion field. To isolate the key underlying physical mechanisms, it is useful to simplify the situation as much as possible. With this in mind, we consider a situation where the back-action of the fermionic fields on the condensate is negligible. Specifically, we assume that $gn_B >> \sum_\sigma f_\sigma n_\sigma$ is satisfied, in which case we can ignore the effects of the fermionic beam on the condensate

wave function $\phi_B(\mathbf{r})$. We then apply a standard Bogoliubov approach[15] to determine the effect of the fermions on the excitation field $\hat{\xi}(\mathbf{r})$. One readily finds

$$\hat{\xi}(\mathbf{r},t) = \hat{\xi}^{(0)}(\mathbf{r},t) + \frac{1}{i\hbar}\int_0^t dt' \int d^3r' \left[G(\mathbf{r},\mathbf{r}',t-t')\phi_B(\mathbf{r}') - F(\mathbf{r},\mathbf{r}',t-t')\phi_B^*(\mathbf{r}')\right] \sum_\sigma f_\sigma \delta\hat{n}_\sigma. \tag{3}$$

The first term $\hat{\xi}^{(0)}(\mathbf{r},t)$ on the right-hand side describes the free-field vacuum quasi-particle fluctuations in the absence of fermions. It has the familiar form

$$\hat{\xi}^{(0)}(\mathbf{r},t) = \sum_{\mathbf{n}} \left(u_{\mathbf{n}}(\mathbf{r}) e^{-iE_{\mathbf{n}}t/\hbar}\hat{\alpha}_{\mathbf{n}} - v^*_{\mathbf{n}}(\mathbf{r}) e^{iE_{\mathbf{n}}t/\hbar}\hat{\alpha}^\dagger_{\mathbf{n}} \right),$$

with the Bogoliubov quasi-particle operators $\hat{\alpha}_{\mathbf{n}}$ and $\hat{\alpha}^\dagger_{\mathbf{n}}$ satisfying Bose commutation relations. The quasi-particle mode functions $u_n(\mathbf{r})$ and $v_n(\mathbf{r})$, and corresponding energy eigenvalues are determined by the matrix equations

$$\begin{bmatrix} \hat{H}_B & -gn_B \\ gn_B & -\hat{H}_B \end{bmatrix} \begin{bmatrix} u_{\mathbf{n}}(\mathbf{r}) \\ v_{\mathbf{n}}(\mathbf{r}) \end{bmatrix} = E_{\mathbf{n}} \begin{bmatrix} u_{\mathbf{n}}(\mathbf{r}) \\ v_{\mathbf{n}}(\mathbf{r}) \end{bmatrix}. \tag{4}$$

The second term in Eq. (3) is a four-wave mixing process that mixes the condensate with quasi-particles. It is mediated by the density fluctuations of the Fermi fields, whose evolution is governed by the quasi-particle Green's functions

$$G(\mathbf{r},\mathbf{r}',\tau) = \sum_{\mathbf{n}} \left(e^{\frac{-iE_{\mathbf{n}}\tau}{\hbar}} u_{\mathbf{n}}(\mathbf{r})u^*_{\mathbf{n}}(\mathbf{r}') - e^{\frac{iE_{\mathbf{n}}\tau}{\hbar}} v^*_{\mathbf{n}}(\mathbf{r})v_{\mathbf{n}}(\mathbf{r}') \right),$$

with a similar form for $F(\mathbf{r},\mathbf{r}',\tau)$, but with $u^*_{\mathbf{n}}(\mathbf{r}')$ and $v_{\mathbf{n}}(\mathbf{r}')$ replaced by $v^*_{\mathbf{n}}(\mathbf{r}')$ and $u_{\mathbf{n}}(\mathbf{r}')$, respectively.

The lowest-order contribution of the condensate to the dynamics of the Fermi fields is obtained by substituting Eq. (3) into Eq. (2). Using the homogeneous case quasi-particle mode functions of Ref. [15] for simplicity, this yields the stochastic Heisenberg equations of motion

$$i\hbar\frac{\partial\hat{\psi}_\sigma}{\partial t} = \left(\hat{H}_{F\sigma} - \mu_\sigma\right)\hat{\psi}_\sigma + h\hat{\psi}^\dagger_{\sigma'}\hat{\psi}_{\sigma'}\hat{\psi}_\sigma + \sum_{\sigma'=\uparrow,\downarrow}\int d^3r' \int_0^t d\tau W_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',\tau)\phi_B(\mathbf{r})\phi_B(\mathbf{r}')\delta\hat{n}_{\sigma'}(\mathbf{r}',t-\tau)\hat{\psi}_\sigma + \hat{\Gamma}_\sigma\hat{\psi}_\sigma. \tag{5}$$

Here we have defined

$$W_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',\tau) = \left(\frac{1}{i\hbar}\right)\left[\Delta_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',\tau) - \Delta^*_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',\tau)\right],$$

$$\Delta_{\sigma\sigma'} = \frac{f_\sigma f_{\sigma'}}{V}\sum_{\mathbf{k}}\sqrt{\frac{\epsilon_{\mathbf{k}}}{\epsilon_{\mathbf{k}}+2gn_B}}e^{-iE_{\mathbf{k}}\tau/\hbar + i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')},$$

with $\epsilon_{\mathbf{k}} = \hbar^2k^2/2m_B$, and $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}}+2gn_B)}$.

The third term in Eq. (5) is the nonlinear optics analogue of a fifth-order nonlinearity, involving two condensate fields, the fermion density fluctuations which are quadratic in the fermion fields, and the fermion field. Since these five fields can mix to produce a sixth, the physical process involved is six-wave mixing between the boson and fermion fields. Finally, the last term in Eq. (5), $\Gamma_\sigma = f_\sigma\phi_B(\mathbf{r})[\xi^{(0)}(\mathbf{r},t) + h.c.]$, is a stochastic potential whose physical origin are the density fluctuations of the vacuum state of the Bose quasi-particles, and has the two-point correlation function $\langle\Gamma_\sigma(\mathbf{r},t)\Gamma_{\sigma'}(\mathbf{r}',t')\rangle = n_B\Delta_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',t-t')$.

To proceed we next assume that the fermion beam propagates through the condensate at velocity v less than the condensate sound velocity $c = \hbar\sqrt{4\pi n_B a_B}/m_B$. Viewing the beam as an impurity traversing the condensate, the condition $v < c$ implies that it will not create incoherent phonon excitations that persist in the condensate after the beam has passed [16]. Physically, in this limit the fermion beam is accompanied as it propagates by a virtual phonon cloud that mediates an effective *instantaneous* interaction between the fermions. Furthermore, since the phonon excitations are virtual the effects of the stochastic potential $\Gamma_\sigma(\mathbf{r},t)$ may safely be neglected. Thus, we may neglect time retardation in the collision term and the stochastic potential in the fermionic field Eq. (5). It then reduces to

$$i\hbar\frac{\partial\hat{\psi}_\sigma(\mathbf{r},t)}{\partial t} = \left(\hat{H}_{F\sigma} - \mu_\sigma\right)\hat{\psi}_\sigma + h\hat{\psi}^\dagger_{\sigma'}\hat{\psi}_{\sigma'}\hat{\psi}_\sigma + \sum_{\sigma'=\uparrow,\downarrow}\int d^3r' U_{\sigma\sigma'}(\mathbf{r},\mathbf{r}')\delta\hat{n}_{\sigma'}(\mathbf{r}',t)\hat{\psi}_\sigma(\mathbf{r}',t). \tag{6}$$

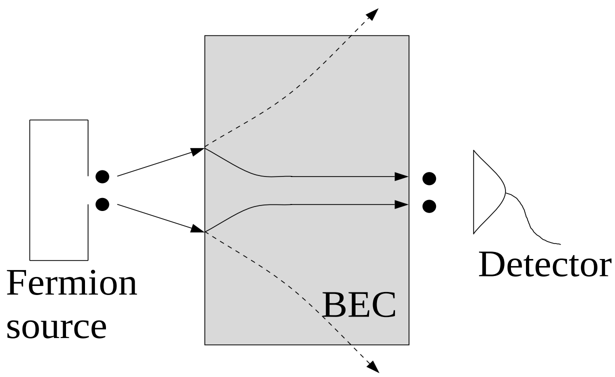

FIG. 1: Schematic diagram for the formation of the two-atom bound state in a fermionic beam propagating through a Bose-Einstein condensate

where for the case of a homogeneous condensate $U_{\sigma\sigma'}(\mathbf{r},\mathbf{r}')$ is the Yukawa potential [8, 9]

$$U_{\sigma\sigma'}(\mathbf{r},\mathbf{r}') = -\left(\frac{m_B f_\sigma f_{\sigma'} n_B}{\pi\hbar^2}\right)\frac{e^{-\sqrt{2}|\mathbf{r}-\mathbf{r}'|/l_h}}{|\mathbf{r}-\mathbf{r}'|} \tag{7}$$

and $l_h = \hbar/\sqrt{2m_B g n_B}$ is the healing length of the Bose condensate. The minus sign in Eq. (7) indicates that the Fermi-Bose coupling induces an *attractive* effective force between fermionic atoms. In the language of nonlinear optics the condensate acts as a "nonlinear crystal" for the fermions and the induced effective fermion-fermion interaction $U_{\sigma\sigma'}(\mathbf{r},\mathbf{r}',t)$ plays the role of a spatially nonlocal Kerr nonlinearity for the fermionic field, leading to familiar effects such as self-focusing. However, since fermionic fields are intrinsically multimode and are not amenable to a mean-field description [17] one cannot expect simple, few-mode nonlinear effects such as occur in optics or bosonic atom optics, but rather multimode coupling and many-particle quantum correlations. Here we consider the particular case of formation of two-fermion bound states, a problem that presents a close analogy to the quantum propagation of a two-photon light beam in a self-focusing medium [18].

Figure 1 shows a potential nonlinear atom optics scheme to realize the bound states. We assume that an ultracold source of fermionic atoms generates a beam containing simultaneously two atoms. First we consider a spin-polarized beam described by the two-atom quantum state, $|\Phi(t)\rangle = \frac{1}{\sqrt{2}}\int d^3r_1\int d^3r_2 f(\mathbf{r}_1,\mathbf{r}_2,t)\hat{\psi}^\dagger(\mathbf{r}_1)\hat{\psi}^\dagger(\mathbf{r}_2)|0\rangle$, that then propagates at velocity v into a condensate. The function $f(\mathbf{r}_1,\mathbf{r}_2,t)$ gives the probability amplitude to find one atom at $\mathbf{r}_1$ and the other at $\mathbf{r}_2$, and is antisymmetric under atomic exchange. (We ignore the spin label for simplicity of notation.) The time evolution of $|\Phi(t)\rangle$ is determined by the Schrödinger equation

$$i\hbar\frac{\partial|\Phi\rangle}{\partial t} = H_{\text{eff}}|\Phi\rangle \tag{8}$$

where the total effective Hamiltonian, constructed from Eq.(6), is

$$H_{\text{eff}} = \int d^3r\hat{\psi}^\dagger(\mathbf{r})\left(H_F - \mu_F\right)\hat{\psi}(\mathbf{r}) + \frac{1}{2}\iint d^3r d^3r' U(\mathbf{r},\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r})\delta\hat{n}(\mathbf{r}')\hat{\psi}(\mathbf{r}). \tag{9}$$

This yields the equation of motion for $f(\mathbf{r}_1,\mathbf{r}_2,t)$

$$i\hbar\frac{\partial f(\mathbf{r}_1,\mathbf{r}_2,t)}{\partial t} = \left(H_F(\mathbf{r}_1) + H_F(\mathbf{r}_2) + U(\mathbf{r}_1,\mathbf{r}_2) + 2\iint d^3r' d^3r'' \left[U(\mathbf{r}_1,\mathbf{r}') + U(\mathbf{r}_2,\mathbf{r}')\right]|f(\mathbf{r}',\mathbf{r}'',t)|^2\right) f(\mathbf{r}_1,\mathbf{r}_2,t), \tag{10}$$

which clearly shows the role of the effective Kerr medium on the fermionic field. To determine the condition for formation of the two-atom bound states of Eq. (10) we transform into the center of mass coordinates $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and make the ansatz $f(\mathbf{r}_1,\mathbf{r}_2,t) = e^{i\mathbf{K}\cdot\mathbf{R}-iEt/\hbar}W(\mathbf{r})$, which leads finally to the eigenvalue problem for the relative motion of two particles in a Yukawa potential

$$\left(-\frac{\hbar^2\nabla_r^2}{2\mu} - \frac{U_0}{r}e^{-\sqrt{2}r/l_h}\right)W(\mathbf{r}) = E_r W(\mathbf{r}), \tag{11}$$

where $U_0 \equiv f^2 m_B n_B/(\pi\hbar^2)$ is the potential strength, $\mu = m_F/2$ the reduced mass, and E_r the energy of relative motion. From Eq.(11), one can see that the spatial

width of the energy eigenstates is dictated by the length scale $l_0 = \hbar^2/(\mu U_0)$. The energy eigenvalues of a Yukawa potential have been extensively investigatetd. An accurate formula [19] has been provided for the number of bound states for a state with total angular momentum quantum number l,

$$\nu = (\sqrt{Z} - \sqrt{Z_l})S_l + 1, \tag{12}$$

where $Z = l_h/(\sqrt{2}l_0)$, $Z_l = 0.8399(1+2.7359l+1.6242l^2)$ and $S_l = 1.1335(1+0.0191l-0.001684l^2)$. Considering the requirement of symmetry of the spatial wave function for the spin-polarized beam, the angular momentum must be $l = 1, 3, 5....$ The condition that there exists at least one bound state is $Z \geq Z_1$. This determines a spatial range $l_0 \leq l_b \equiv l_h/6.366$ for two-atom bound state in a Bose condensate, and meanwhile enforces a requirement of sufficient potential strength $U_0 \geq U_b \equiv 6.366\hbar^2/(\mu l_h)$ for binding.

Possible candidates to observe two-fermion bound states include combinations of alkali atoms such as ^{6}Li-^{7}Li, ^{6}Li-^{23}Na, and ^{40}K-^{39}K. The ^{40}K-^{39}K combination appears to be the most promising, due to its very large boson-fermion scattering length $a_{BF} \sim 52.9$ nm and its very small boson-boson scattering length $a_B \sim 0.26$ nm. In this case, the realization of bound fermionic states requires a condensate density of $n_B \sim 10^{13}$ cm^{-3}, a value achievable with current techniques.

The basic signature of the formation of a two-fermion bound state in the condensate is that it is immune to wave packet spreading and can therefore be collected at a localized detector placed at the output of the condensate as illustrated in Fig. 1. In contrast, if the condensate is absent, and the time of flight of the fermions is chosen long enough, the initial fermion packet will spread out such as to negligibly overlap the detector. The signature of the formation of bound states in this nonlinear atom optics arrangement is therefore an enhanced detection of atoms when the condensate is in place in contrast to when it is absent. This remains true even though the initial fermion wave packet does not exactly match the bound state, as long as it has some reasonable projection onto the bound state wave function.

We remark that our results are readily generalized to fermionic atomic beam with two internal spin states. In particular, if one initially prepares the beam in a spin-singlet state, the two-atom quantum state will have the form $|\Phi\rangle = \frac{1}{2}\iint d^3r_1 d^3r_2 f(\mathbf{r}_1, \mathbf{r}_2, t)(\psi^\dagger_\uparrow(\mathbf{r}_1)\psi^\dagger_\downarrow(\mathbf{r}_2) - \psi^\dagger_\downarrow(\mathbf{r}_1)\psi^\dagger_\uparrow(\mathbf{r}_2))|0\rangle$. Then in addition to the attractive Yukawa potential the two fermions also experience an s-wave interaction, since they are in different spin states. Nonetheless two-fermion bound states can still be formed under suitable combination of these potentials. In particular, the formation of the spatial bound states in the spin-singlet case may be useful to create the spatially entangled fermionic atomic beam.

In conclusion, we have derived a nonlinear Schrödinger equation for a fermionic beam in a condensate, and shown that the condensate acts as an effective nonlinear medium for the fermions, leading to formation of a two-atom bound state closely analogous to the two-photon bound state in self-focusing media [18]. A clear signature of the bound state in a nonlinear atom optics experiment has also been proposed. Physically, the understanding of fermionic bound states may be important for the manipulation of the quantum statistical properties of fermionic atomic beams, e.g. changes from antibunched and bunched beams, dynamic Cooper pairing, and potentially the formation of quantum solitons in ultracold fermionic atomic beams.

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[1] B. DeMarco and D. S. Jin, Science **285**, 1703 (1999).
[2] A. G. Truscott *et al.*, Science **291**, 2570 (2001).
[3] F. Schreck *et al.*, Phys. Rev. Lett. **87**, 080403 (2001).
[4] S. R. Granade *et al.*, Phys. Rev. Lett. **88**, 120405 (2002).
[5] Z. Hadzibabic,*et al.*, Phys. Rev. Lett. **88**, 160401 (2002).
[6] K. Mølmer, Phys. Rev. Lett. **80**, 1804 (1998).
[7] M. Amoruso *et al.*, Eur. Phys. J. D **4**, 261 (1998).
[8] M. J. Bijlsma, B. A. Heringa and H. T. C. Stoof, Phys. Rev. A **61**, 053601 (2000).
[9] L. Viverit *et al.*, Phys. Rev. A **61**, 053605 (2000).
[10] S. K. Yip, Phys. Rev. A **64**, 023609 (2001).
[11] H. Pu *et al.*, Phys. Rev. Lett. **88**, 070408 (2002).
[12] H. T. C. Stoof *et al.*, Phys. Rev. Lett. **76**, 10 (1996); M. Houbiers *et al.*, Phys. Rev. A **56**, 4864 (1997); G. Bruun *et al.*, Eur. Phys. J. D **7**, 433 (1999); J. L. Bohn, Phys. Rev. A **61**, 053409 (2000); T. Loftus *et al.*, e-print cond-mat/0111571; M. Mackie *et al.*, Opt. Express **8**, 118 (2000); M. Mackie *et al.*, e-print physics/0104043; E. Timmermans *et al.*, Phys. Lett. A **285**, 228 (2001); M. Holland *et al.*, Phys. Rev. Lett. **87**, 120406 (2001).
[13] M. G. Moore and P. Meystre, Phys. Rev. Lett. **86**, 4199 (2001).
[14] W. Ketterle and S. Inouye, Phys. Rev. Lett. **86**, 4203 (2001).
[15] A. L. Fetter, Annals of Physics **70**, 67-101 (1972).
[16] E. Timmermans and R. Cote, Phys. Rev. Lett. **80**, 3419 (1998).
[17] An exception is that of fermionic atoms that form Cooper pairs after undergoing a BCS phase transition. The Cooper pairs can be described using the mean-field Ginzburg-Landau theory.
[18] R. Y. Chiao and I. H. Deutsch, Phys. Rev. Lett. **67**, 1399 (1991).
[19] A. E. S. Green, Phys. Rev. A **26**, 1759(1982).